Radiative Transfer Within a Cylindrical Geometry with Fresnel Reflecting Boundary

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Formal integral equations are presented for the radiative heat transfer within an emitting, absorbing, and isotropically scattering cylindrical medium with specular reflection and axisymmetric radiative load at boundary and with axisymmetric temperature distribution. Expressions are then used for the solutions of the directional emittance from cylindrical media with Fresnel reflectivities on the boundary at various optical diameters and various values of albedo and refractive index. The problem is solved using the method of weighting functions. The planar presentation of conic results is given at various azimuthal angles. Results show that the maximum directional emittance occurs at a higher refractive index as albedo increases. Results also show the scattering and dimensional effects.

Nomenclature

C_i	= expansion coefficients
I	= radiation intensity
n	= relative refractive index
N	= order of solution
r	= radial coordinate
R	= optical radius
s'	= dummy variable
S	= source function
$\mathcal{S}$	= dimensionless source function ($= S/n^2 I_b[T]$)
T	= medium temperature
z	= axial coordinate
τ_d	= optical distance between an interior point and a boundary point
τ_o	= optical distance between two boundary points
β	= extinction coefficient
ω	= albedo
θ	= polar angle
ϕ	= azimuthal angle
ρ	= interface reflectivity
ϵ	= directional emittance
ν	= frequency
μ	= cosine of incident angle
π	= pi
Subscripts	
b	= blackbody quantity
o	= quantity at the boundary
in	= entrant radiation

Introduction

THE radiation heat transfer in solids is a problem of considerable technical importance.[1] Several studies have been performed to examine the influence of refractive index on the emission characteristics of a scattering medium.[2-9] Turner and Love presented the directional emissivity from infinite and semi-infinite slabs of scattering media having various values of refractive index, using the time-consuming Monte-Carlo statistical method.[2,3] Armaly et al. presented the emissivities of a semi-infinite medium with the use of the exponential kernel substitution technique.[4] The same technique was also utilized to analyze a finite medium.[5] Crosbie formulated the emittance of semi-infinite and finite media in terms of Chandrasekhar's eigenfunctions and obtained exact numerical results.[6,7] Lin et al. studied a finite medium using the source function expansion technique.[8] Recently, Wu utilized the image method to formulate the emittance of a semi-infinite slab.[9]

The radiant transport study reported here is concerned with a cylindrical geometry. The medium emits, absorbs, and isotropically scatters thermal radiation. Fresnel relations are used to evaluate the reflection and transmission at the boundary. One of the purposes of this study is to develop exact expressions for radiant transport in a cylindrical geometry with axisymmetric temperature distribution and external radiation axisymmetrically irradiating the cylinder boundary. Expressions are then used for the prediction of the directional emittance. The Galerkin method, which has been applied to an infinite slab,[10] is used to solve the integral form of the equation of radiative transfer. This numerical analysis is intended to show the effects of optical radius, refractive index, and scattering albedo as well as the effect of geometry on the directional emittance of a cylinder.

Formulation

The system considered is an infinitely long cylindrical medium with an axisymmetric radiative load at the boundary and axisymmetric temperature distribution, $T(r)$. Assume that the medium is isotropic and emits, absorbs, and isotropically scatters thermal radiation. The medium is characterized by a single scattering albedo ω, optical radius R, and relative refractive index n which is the ratio of refractive index of cylinder to that of the surroundings. The geometry and coordinate system are shown in Fig. 1. Fresnel reflectivities are included at the boundary. In the following, the subscript ν, which denotes the spectral-dependent properties of the medium, is omitted for simplicity.

Let $S(r)$ represent the source function at location r, which consists of the emitted and scattered radiation. Then, the source function distribution can be obtained by[11]

$$S(r) = (1-\omega)n^2 I_b[T(r)] + \frac{\omega}{2\pi}\left\{\int_0^R\int_0^{2\pi} S(r')K_1\left(\sqrt{r^2+r'^2-2rr'\cos\theta}\right) \times r'\mathrm{d}\theta\mathrm{d}r' + \int_0^{2\pi}\int_0^{\infty} I_o \frac{e^{-\tau_d}}{\tau_d^3} R(R - r\cos\theta)\,\mathrm{d}z\mathrm{d}\theta\right\} \quad (1)$$

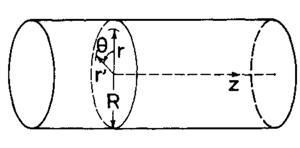

Fig. 1 Schematic diagram and coordinates for a cylinder.

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where $\tau_d = \sqrt{z^2 + r^2 + R^2 - 2rR\cos\theta}$ and I_o is the intensity of inward radiation at the boundary point (R,θ,z) in the direction of the vector from (R,θ,z) to the point $(r,0,0)$. $IK_n(y) = Ki_n(y)/y^n$ and $Ki_n(y)$ are the nth integral of the modified Bessel function of order zero, i.e.,

$$Ki_n(y) = \int_l^\infty \frac{e^{-yt}}{t^n\sqrt{t^2-1}}dt \tag{2}$$

The boundary intensity I_o consists of the reflected and entrant radiation and can be obtained by the ray tracing technique as

$$I_o = \rho(\mu)\left[\int_0^x S(\bar{r}')e^{-s'}ds' + I_o e^{-x}\right] + \left(I_{in}\right) \tag{3}$$

or

$$I_o = \left[\rho(\mu)\int_0^x S(\bar{r}')e^{-s'}ds' + I_{in}\right]\Big/\left[1 - \rho(\mu)e^{-x}\right] \tag{4}$$

where I_{in} is the intensity of entrant radiation at the boundary point (R,θ,z) in the same direction of I_o, $\mu = (R - r\cos\theta)/\tau_d$ is the cosine of incident angle, $x = 2R(R - r\cos\theta)\tau_d/(r^2 + R^2 - 2rR\cos\theta)$,

$$\bar{r}' = \sqrt{R^2 - 2R(R - r\cos\theta)\frac{s'}{\tau_d} + (r^2 + R^2 - 2r\cos\theta)\left(\frac{s'}{\tau_d}\right)^2}$$

and $\rho(\mu)$ is the interface reflectance.

ϕ Substituting Eq. (4) for Eq. (1), we have

$$S(r) = (1-\omega)n^2 I_b[T(r)] + \frac{\omega}{2\pi}\left\{\int_0^R\int_0^{2\pi} S(r')IK_1\left(\sqrt{r^2 + r'^2 - 2rr'\cos\theta}\right)\right.$$

$$\times r'd\theta dr' + \int_0^{2\pi}\int_0^\infty \frac{I_{in} + \rho(\mu)\int_0^x S(r')e^{-s'}ds'}{1 - \rho(\mu)e^{-x}}$$

$$\left.\times \frac{e^{-\tau_d}}{\tau_d^3}R(R - r\cos\theta)\,dz\,d\theta\right) \tag{5}$$

Equation (5) formulates the radiative transfer of the axisymmetric problem having a specularly reflecting boundary. Once the source function distribution is solved, the radiant flux and intensity can be obtained in terms of the source function.

Directional Emittance

Equation (5) may be utilized for the prediction of directional emittance of a cylindrical medium. For a medium with uniform temperature distribution and no entrant radiation, the dimensionless source function, which is defined as the ratio $S/[n^2 I_b(T)]$, is given by

$$\$(r) = (1-\omega) + \frac{\omega}{2\pi}\left\{\int_0^R\int_0^{2\pi} \$(r')\mathrm{IK}_1\left(\sqrt{r^2 + r'^2 - 2rr'\cos\theta}\right)\right.$$

$$\times r'd\theta dr' + \int_0^{2\pi}\int_0^\infty\int_0^x \frac{\rho(\mu)\$(\bar{r}')e^{-s'}}{1 - \rho(\mu)e^{-x}}$$

$$\left.\times \frac{e^{-\tau_d}}{\tau_d^3}R(R - r\cos\theta)ds'dz\,d\theta\right\} \tag{6}$$

where x, μ, $\bar{r}'$, and τ_d are as defined earlier. The directional emittance of such a cylindrical medium is then given in terms of dimensionless source function as

$$\mathcal{E}(\theta', \phi) = \frac{[1 - \rho(\cos\bar{\theta})]I(\bar{\theta}, \phi)}{n^2 I_b[T]}$$

$$= \frac{1 - \rho(\cos\bar{\theta})}{1 - \rho(\cos\bar{\theta})e^{-\tau_o}}\int_0^{\tau_o}\$(\bar{r})e^{-s'}ds' \tag{7}$$

where $\bar{\theta}$ is related to the escape polar angle θ' from Snell's law by $\sin\theta' = n\cdot\sin\bar{\theta}$, and ϕ is the azimuthal angle, as illustrated in Fig. 2. $\tau_o = 2R\cos\bar{\theta}/(\cos^2\bar{\theta} + \sin^2\bar{\theta}\sin^2\phi)$ and $\bar{r} = [R^2 - 2R\cos\bar{\theta}s' + (\cos^2\bar{\theta} + \sin^2\bar{\theta}\cdot\sin^2\phi)s'^2]^{1/2}$.

It is shown that the expression for $I(\bar{\theta}, \phi)$ in Eq. (7) is identical to that of Mitsis in terms of $\bar{\theta}$ and $\bar{\Phi}$.[12]

The reflectance of smooth interface can be calculated by Fresnel relations, i.e.,

$$\rho(\mu) = \begin{cases} 1, & 0 \le \mu \le \sqrt{1 - 1/n^2} \\ \frac{1}{2}\left[\frac{n\mu - \sqrt{1 - n^2(1-\mu^2)}}{n\mu + \sqrt{1 - n^2(1-\mu^2)}}\right]^2 & \\ \times\left\{1 + \left[\frac{\mu\sqrt{1-n^2(1-\mu^2)} - n(1-\mu^2)}{\mu\sqrt{1-n^2(1-\mu^2)} + n(1-\mu^2)}\right]^2\right\}, & \sqrt{1-1/n^2} \le \mu \le 1 \end{cases} \tag{8}$$

To obtain the directional emittance, we first have to determine the source function distribution. Since a direct analytic solution to the integral equation, Eq. (6), is quite difficult to obtain, we apply the Galerkin method which has been successfully applied to the integral form of equation for incident radiation and shows straightforward in Ref. 10. We consider a power series representation of the function $\$(r)$ as

$$\$(r) = \sum_{i=0}^{N} C_i\left(\frac{r}{R}\right)^i \tag{9}$$

where C_i are the unknown expansion coefficients which are to be determined, and N is the order of the approximation.

The application of the Galerkin method yields $N+1$ algebraic equations for the determination of the expansion coefficients. These algebraic equations can be written as

$$\sum_{i=0}^{N} B_{m,i}C_i = A_m,\ m = 0, 1, 2, \ldots, N \tag{10}$$

where

$$A_m = \frac{1-\omega}{m+1}R$$

$$B_{m,i} = \frac{R}{m+i+1} - \frac{\omega}{2\pi}\left\{\int_0^R\int_0^R\int_0^{2\pi}\left(\frac{r'}{R}\right)^i\left(\frac{r}{R}\right)^m\right.$$

$$\times \mathrm{IK}_1\left(\sqrt{r^2 + r'^2 - 2rr'\cos\theta}\right)r'd\theta dr'dr$$

$$+ \int_0^R\int_0^{2\pi}\int_0^\infty\int_0^x \frac{\rho(\mu)\left(\frac{\bar{r}'}{R}\right)^i e^{-s'}}{1 - \rho(\mu)e^{-x}}\left(\frac{r}{R}\right)^{m} \frac{e^{-\tau d}}{\tau_d^3}$$

$$\left.\times R(R - r\cos\theta)ds'dz\,d\theta dr\right\} \tag{11}$$

Once the expansion coefficients are determined from Eq. (10), then the directional emittance is readily obtained by Eq. (7).

Results

The accuracy of the Galerkin method has been analyzed in Ref. 10. It was seen that less order of the approximation could give very accurate solutions as albedo increases.[10] Since no exact solutions exist for the present problem, we compare, in Table 1, the convergence for $N = 2$ and $N = 3$ at optical radius $R = 1.0$, albedo $\omega = 0.2$, and refractive index $n = 1.2$. It is noticed that the solutions of directional emittance at $N = 2$ agree with the $N = 3$ case to at least three digits. The source function is converged to one less digit than the emittance. All results of directional emittance shown in Figs. 3-5 are second order (i.e., three term) approximations. Although the integral over θ in the kernel IK may lead to an expression involving the modified Bessel functions, we evaluate the integrals involved in the definition of $B_{m,i}$ with Gaussian quadratures. The integral over z is obtained by choosing a sufficiently large number instead of infinity.

Figure 3 shows the emittance from a cylindrical geometry with the refractive index of unity at various optical radii for both large and small albedos. At $n = 1$, $\rho(\mu) = 0$ for Fresnel interfaces, i.e., the medium is with nonreflecting free boundary. Compared to the directional emittance of a slab,[8] the present results at $\phi = 0$ deg are only slightly smaller than slab emittance for a small albedo. This is because the scattering contribution is insignificant relative to the emission for small albedo and no reflection occurs at the boundary. When the scattering becomes strong and/or reflectivity is included at the boundary, dimensional and geometrical effects become important on the source function distribution. For such cases, the results for the present geometry show much smaller values for directional emittance than obtained in the

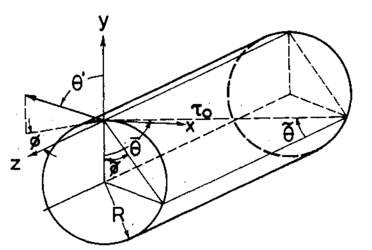

Fig. 2 Local spherical coordinates and leaving and incident polar angles.

Table 1 Influence of the order of the approximation for $R = 1.0$, $n = 1.2$, and $\omega = 0.2$.

	$N = 2$	$N = 3$
	Emittance	
$\phi = 0°$ $\theta' = 0°$	0.79825	0.79838
20°	0.80832	0.80845
40°	0.83406	0.83419
60°	0.84867	0.84879
80°	0.64730	0.64739
$\phi = 30°$ $\theta' = 0°$	0.79825	0.79838
20°	0.80213	0.80231
40°	0.81020	0.81031
60°	0.80142	0.80125
80°	0.60758	0.60732
$\phi = 60°$ $\theta' = 0°$	0.79825	0.79838
20°	0.79018	0.79034
40°	0.76506	0.76484
60°	0.70973	0.70939
80°	0.52450	0.52444
$\phi = 90°$ $\theta' = 0°$	0.79825	0.79838
20°	0.78435	0.78447
40°	0.74343	0.74313
60°	0.66785	0.66760
80°	0.48743	0.48751
	Source Function	
$r = 0.019855$	0.95862	0.95666
0.101667	0.95416	0.95402
0.237234	0.94675	0.94787
0.408283	0.93735	0.93803
0.591717	0.92720	0.92653
0.762766	0.91768	0.91657
0.898333	0.91009	0.91026
0.980145	0.90549	0.90749

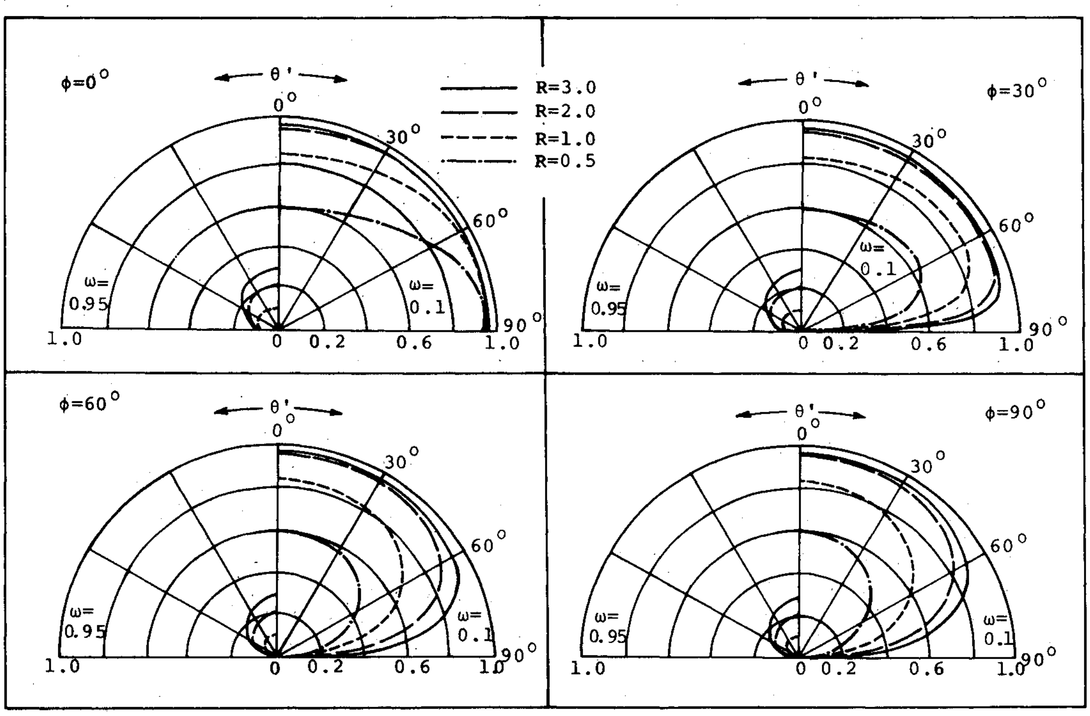

Fig. 3 Directional emittance of a cylinder with refractive index of unity.

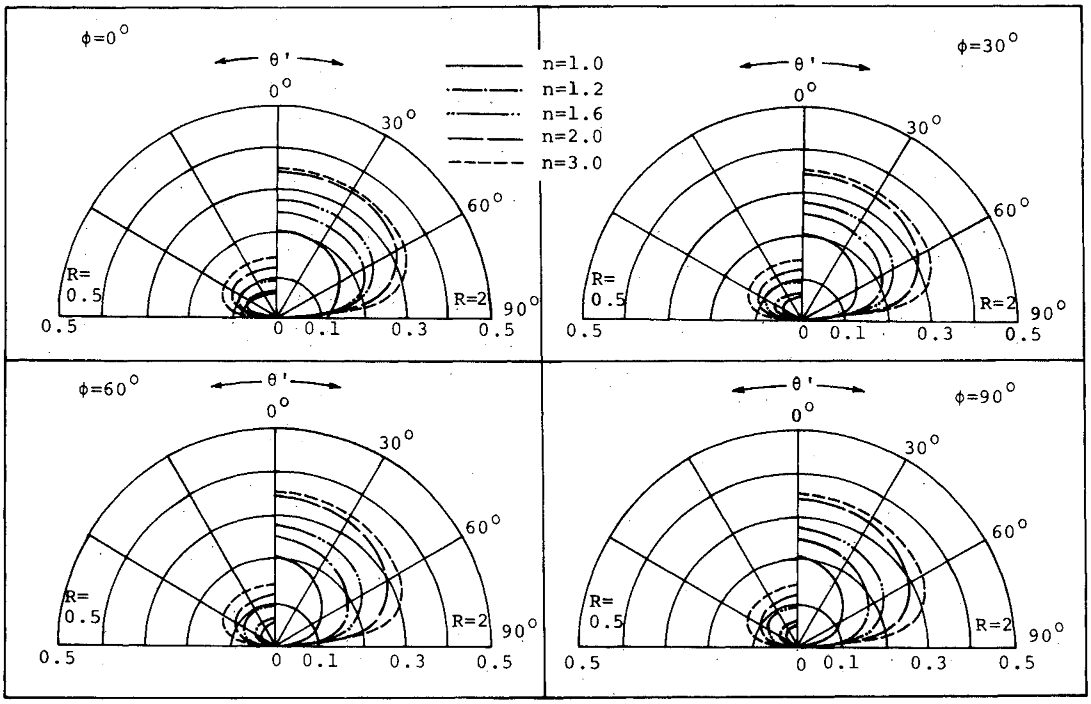

Fig. 4 Variation of directional emittance with the refractive index at $\omega = 0.1$.

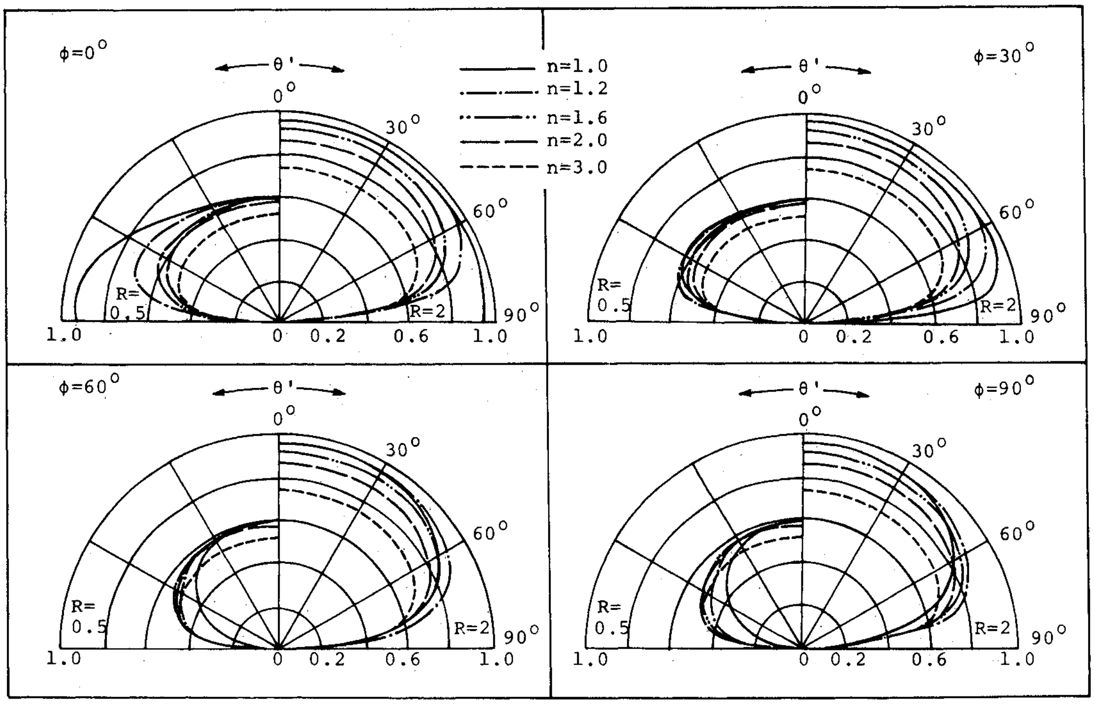

Fig. 5 Variation of directional emittance with the refractive index at $\omega = 0.95$.

slab analysis. The azimuthal dependency of the emittance is also shown in Fig. 3, as well as in the following.

Figures 4 and 5 show the variation of the directional emittance from a cylinder, with the change in refractive index for two values of albedo at $R = 0.5$ and 2.0 This is consistent with the results in Refs. 6 and 8, which show that the refractive index at which the maximum emittance occurs increases with albedo. It is also seen that the azimuthal dependency of the emittance gradually vanishes with the increase in the refractive index. At $n = 2.0$ and 3.0, the results are nearly independent of the azimuthal angle. This is due to the fact that the angular region in the medium over which the source function distribution contributes to the escaping radiation becomes smaller as the refractive index increases. Figures 4 and 5 also show the effect of albedo on the directional emittance. The directional emittance increases with the decrease in albedo and with the increase in optical radius.

Summary

This paper presents exact integral expressions for the radiative transfer within an emitting, absorbing, and isotropically scattering cylindrical medium having specularly reflecting boundary and having axisymmetric radiative load and axisymmetric temperature distribution.

The influence of the refractive index on the emission characteristics of a cylindrical medium with uniform temperature is analyzed with the use of the Galerkin method. The numerical analysis leads to the following conclusions:

(1) This method provides a very straightforward approach for the solution of the problem.

(2) As the refractive index increases, the directional emittance become less dependent on the azimuthal angle.

(3) The refractive index at which the maximum emittance occurs increases with albedo.

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